

THE HEISENBERG COBOUNDARY EQUATION: APPENDIX TO *EXPLICIT CHABAUTY-KIM THEORY*

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ABSTRACT. Let p be a regular prime number, let $G_{\{p\}}$ denote the Galois group of the maximal unramified away from p extension of \mathbb{Q} , and let $H_{\text{ét}}$ denote the Heisenberg group over \mathbb{Q}_p with $G_{\{p\}}$ -action given by $H_{\text{ét}} = \mathbb{Q}_p(1)^2 \oplus \mathbb{Q}_p(2)$. Although Soulé vanishing guarantees that the map $H^1(G_{\{p\}}, H_{\text{ét}}) \rightarrow H^1(G_{\{p\}}, \mathbb{Q}_p(1)^2)$ is bijective, the problem of constructing an explicit lifting of an arbitrary cocycle in $H^1(G_{\{p\}}, \mathbb{Q}_p(1)^2)$ proves to be a challenge. We explain how we believe this problem should be analyzed, following an unpublished note by Romyar Sharifi, hereby making the original appendix to *Explicit Chabauty-Kim theory* available online in an arXiv-only note.

1. THE CONTEXT

This brief note began its life as an appendix to *Explicit Chabauty-Kim theory for the thrice punctured line in depth two* [DCW], which received an appendectomy prior to publication. Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, let

$$S = \{q_1, \dots, q_s\}$$

denote a finite set of primes, let

$$\mathbf{S} = \text{Spec } \mathbb{Z} \setminus S,$$

let p denote a prime $\notin S$, and let $T = S \cup \{p\}$. Kim's approach to the study of the set $X(\mathbf{S})$ of S -integral points of X involves a certain tower of morphisms of affine finite-type \mathbb{Q}_p -varieties. As we explain in *Explicit Chabauty-Kim theory*, its first two steps look like so.

$$\begin{array}{ccc} H_f^1(G_T, H_{\text{ét}}) & \xrightarrow{h_2} & \mathbb{A}_{\mathbb{Q}_p}^3 \\ \cong \downarrow \pi_* & & \downarrow \\ H_f^1(G_T, \mathbb{Q}_p(1)^2) & \xrightarrow{h_1} & \mathbb{A}_{\mathbb{Q}_p}^2 \end{array}$$

Here G_T denotes the Galois group of the maximal unramified outside of T extension of \mathbb{Q} , $H_{\text{ét}}$ denotes the Heisenberg group object with G_T -action given simply by $H_{\text{ét}} = \mathbb{Q}_p(1)^2 \oplus \mathbb{Q}_p(2)$, and the H_f^1 's are certain subschemes of nonabelian cohomology varieties. The map induced by abelianization

$$\pi : H_{\text{ét}} \rightarrow \mathbb{Q}_p(1)^2$$

on the level of H_f^1 's fits in a commuting square like so.

$$\begin{array}{ccc}
H_f^1(G_T, H_{\text{ét}}) & \hookrightarrow & H^1(G_T, H_{\text{ét}}) \\
\cong \downarrow \pi_* & & \cong \downarrow \pi_* \\
H_f^1(G_T, \mathbb{Q}_p(1)^2) & \hookrightarrow & H^1(G_T, \mathbb{Q}_p(1)^2) \\
\parallel & & \parallel \\
\mathbb{Q}_p^S & \hookrightarrow & \mathbb{Q}_p^T
\end{array}$$

Our main result in *Explicit Chabauty-Kim theory* is a complete computation of the map

$$h_2 \circ \pi_*^{-1} : \mathbb{Q}_p^S \rightarrow \mathbb{Q}_p^3.$$

Since our methods there were somewhat indirect, we document here our initial attempt to compute π_*^{-1} directly.

2. THE PROBLEM

2.1. By Corollary 6.2.3 of [DCW], an A -point of $H_f^1(G_T, \mathbb{Q}_p(1))$ may be written κ_x , where

$$x = q_1^{x_1} \cdots q_s^{x_s}$$

is a formal product of powers, with $x_1, \dots, x_r \in A$. If

$$y = q_1^{y_1} \cdots q_s^{y_s}$$

denotes another point, the cup product $\kappa_y \cup \kappa_x$ is an element of $Z^1(G_T, A(2))$. For simplicity, restrict attention to the case $A = \mathbb{Q}_p$, and consider the cochain complex

$$0 \rightarrow C^0(\mathbb{Q}_p(2)) \rightarrow C^1(\mathbb{Q}_p(2)) \rightarrow C^2(\mathbb{Q}_p(2)) \rightarrow C^3(\mathbb{Q}_p(2)) \rightarrow \cdots$$

for the cohomology of G_T with coefficients in $\mathbb{Q}_p(2)$. By the vanishing results

$$H^1(G_T, \mathbb{Q}_p(2)) = H^2(G_T, \mathbb{Q}_p(2)) = 0,$$

the equation

$$\kappa_y \cup \kappa_x = d\alpha$$

in C^2 admits a solution $\alpha \in C^1$, unique up to translation by the coboundary $d\beta : \sigma \mapsto \beta - \sigma(\beta)$ of an element $\beta \in \mathbb{Q}_p(2)$. Recalling the definition of the cup product and the second coboundary, we have

$$\kappa_x(\sigma) \otimes \sigma \kappa_y(\tau) = \alpha(\sigma\tau) - \sigma\alpha(\tau) - \alpha(\sigma).$$

Here, σ and τ vary over G_T . We call this *the Heisenberg coboundary equation*.

2.2. By §5 of [DCW], we have an exact sequence

$$1 \longrightarrow \mathbb{Q}_p(2) \longrightarrow H_{\text{ét}} \xrightleftharpoons{\Sigma} \mathbb{Q}_p(1)^2 \longrightarrow 1$$

of nonabelian G_T -modules, which is split if we identify $H_{\text{ét}}$ with its Lie algebra and forget the bracket. Fix a point $(x, y) \in (\mathbb{Z}[S^{-1}]^* \otimes \mathbb{Q}_p)^2$ in the source of the unipotent p -adic Hodge morphism in depth one. We then have associated Kummer cocycles $\kappa_x, \kappa_y : G_T \rightrightarrows \mathbb{Q}_p(1)$, and by composing with Σ , we obtain a candidate $\Sigma(\kappa_x, \kappa_y) \in C^1(G_T, H_{\text{ét}})$ for a lifting of (κ_x, κ_y) to depth two. By segment 2.3.2 of [DCW], its failure to be a cocycle is measured by a solution α of the Heisenberg coboundary equation. So if we set $\kappa_{x,y} := \alpha^{-1} \cdot \Sigma(\kappa_x, \kappa_y)$, we obtain a representative for the element of $H^1(G_T, H_{\text{ét}})$ which maps to (κ_x, κ_y) in $H^1(G_T, \mathbb{Q}_p(1)^2)$.

2.3. The problem then, is to make the solution α of the Heisenberg coboundary equation in some way explicit.

3. STEPS TOWARDS ITS SOLUTION

3.1. Soulé's proof of the vanishing of $H^2(\mathbb{Q}_p(2))$ is not well adapted to this application. A simpler proof is given by Romyar Sharifi in an unpublished note [Sha], for the case $p = 2$. As Sharifi points out, the essential property of the even prime which makes his proof possible is its regularity. Sharifi's proof goes roughly as follows. Let K_T denote the maximal unramified outside T extension of \mathbb{Q} . Let $G_{T,n}$ denote the Galois group of K_T over $\mathbb{Q}(\zeta_{p^n})$. Similarly, we let $G_{T,\infty}$ denote the Galois group of K_T over $\mathbb{Q}(\zeta_{p^\infty})$. Noting that the Galois group of $\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}$ is $(\mathbb{Z}/p^n)^*$, we have the following tower of fields and Galois groups for each n .

$$\begin{array}{ccc} & K_T & \\ & \downarrow & \\ G_{T,n} & | & \\ & \mathbb{Q}(\zeta_{p^n}) & \\ (\mathbb{Z}/p^n)^* & | & \\ & \mathbb{Q} & \end{array} \quad \begin{array}{c} \curvearrowright \\ G_T \end{array}$$

By direct computation applied to the low degree terms of the Hochschild-Serre spectral sequence, we obtain an isomorphism

$$(\star) \quad H^1(G_T, \mathbb{Q}_p(2)) = H^1(G_{T,\infty}, \mathbb{Q}_p(2))^{\mathbb{Z}_p^*}.$$

On the other hand, for p regular, we have

$$(\star\star) \quad \mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\mathbb{Z}[T^{-1}, \zeta_{p^\infty}]^*_{/p})(1) = H^1(G_{T,\infty}, \mathbb{Q}_p(2)).$$

The subscript $/p$ indicates p -adic completion. The argument here may be summarized as follows: there's always an injection from the left to the right coming from the Kummer exact sequence; the cokernel lives inside the Picard group (suitably interpreted), whose (pro-)order is (pro-)coprime to p . A study of the action of \mathbb{Z}_p^* on $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\mathbb{Z}[T^{-1}, \zeta_{p^\infty}]^*_{/p})(1)$ now leads to the conclusion that

$$H^1(G_T, \mathbb{Q}_p(2)) = \mathbb{Q}_p.$$

Finally, Poitou-Tate duality is used as a vehicle to get to H^2 .

Actually, throughout most of the proof, Sharifi works with n finite. For n finite, statements analogous to (\star) , $(\star\star)$ fail. Their failure however, is measured by groups whose order turns out to be finite and bounded in n .

3.2. Sharifi's use of Poitou-Tate duality presents for us an obstacle. On the other hand, since *many* regular primes are known to exist (see, for instance §5.3 of Washington [Was]), the stipulation that p be regular is relatively harmless. So a possible approach may be to attack the vanishing of H^2 (or at least of the relevant elements of H^2) directly, by methods inspired by Sharifi's computation of H^1 . To do so, we would replace 3.1 (\star) by an analysis of the map

$$(\star) \quad H^2(G_T, \mathbb{Q}_p(2)) \rightarrow H^2(G_{T,\infty}, \mathbb{Q}_p(2))^{\mathbb{Z}_p^*},$$

and we would replace 3.1 $(\star\star)$ by the map

$$(\star\star) \quad \mathbb{Q}_p \otimes_{\mathbb{Z}_p} K_2^M(\mathbb{Z}[\zeta_{p^\infty}, T^{-1}]) \rightarrow H^2(G_{T,\infty}, \mathbb{Q}_p(2)),$$

while keeping track of the \mathbb{Z}_p^* action.

4. AN ENSUING FAMILY OF SPECTRAL SEQUENCES IN GALOIS COHOMOLOGY

4.1. If 3.2(☆) fails to be bijective, the failure is best measured by certain terms in an associated family of spectral sequences. Let

$$1 \rightarrow N \rightarrow G \rightarrow Q \rightarrow 1$$

be a short exact sequence of (topological) groups, and A a (continuous) $\mathbb{Z}[G]$ -module whose addition law we denote by \star . Then the (continuous) cohomology groups $H^q(N, A)$ have a natural structure of (continuous) $\mathbb{Z}[Q]$ -module, and there's a spectral sequence

$$E_2^{p,q} = H^p(Q, H^q(N, A)) \Rightarrow H^{p+q}(G, A) .$$

Elements of $H^1(N, A)$ may be represented by (continuous) maps $\phi : N \rightarrow A$ which satisfy

$$\phi(\sigma\tau) = \phi(\sigma) \star \sigma\phi(\tau) .$$

If ϕ is such a map and α is an arbitrary element of Q , then the action of Q on $H^1(N, A)$ is given in terms of cocycles by lifting α arbitrarily to an element γ of G and declaring that for any $\eta \in N$,

$$\phi^\alpha(\eta) = \alpha^{-1}\phi(\gamma\eta\gamma^{-1}) .$$

See §5, 6 of Chapter VII of [Ser].

4.2. For each $n > m$, we may apply this to the short exact sequence

$$1 \rightarrow G_{T,n} \rightarrow G_T \rightarrow (\mathbb{Z}/p^n)^* \rightarrow 1 ,$$

with coefficients in $(\mathbb{Z}/p^m)(2)$. If we set

$${}_n^m E_2^{p,q} := H^p((\mathbb{Z}/p^n)^*; H^q(G_{T,n}; \mathbb{Z}/p^m(2)))$$

and

$${}_n^m H^r := H^r(G_T, \mathbb{Z}/p^m(2)) ,$$

then there's a spectral sequence

$${}_n^m E_2^{p,q} \Rightarrow {}_n^m H^r .$$

Relevant terms and arrows of this spectral sequence are pictured below.

$$\begin{array}{ccc}
 {}_n^m E_2^{0,2} & & {}_n^m E_3^{0,2} \\
 \searrow & & \searrow \\
 {}_n^m E_2^{1,1} & \rightarrow & {}_n^m E_2^{2,1} \\
 \searrow & & \searrow \\
 {}_n^m E_2^{2,0} & \rightarrow & {}_n^m E_2^{3,0} \quad \quad {}_n^m E_3^{3,0}
 \end{array}$$

4.3. We now discuss the terms ${}_n^m E_2^{2,0}$. We set $n = m$ for simplicity.

Proposition. Each ${}_n^m E_2^{2,0} = H^2((\mathbb{Z}/p^n)^*, H^0(G_{T,n}, \mathbb{Z}/p^n(2)))$ is a finite group of bounded order.

The proof is in segments 4.4–4.6.

4.4. $G_{T,n}$ acts trivially on $\mathbb{Z}/p^n(2)$, so

$$H^0(G_{T,n}, \mathbb{Z}/p^n(2)) = \mathbb{Z}/p^n(2).$$

We have a short exact sequence of groups

$$0 \rightarrow 1 + (p) \rightarrow (\mathbb{Z}/p^n)^* \rightarrow \mathbb{F}_p^* \rightarrow 0$$

from which we obtain a spectral sequence

$$F_2^{p,q} = H^p(\mathbb{F}_p^*, H^q(1 + (p), \mathbb{Z}/p^n(2))) \Rightarrow H^{p+q}((\mathbb{Z}/p^n)^*, \mathbb{Z}/p^n(2)).$$

It suffices to show that the terms $F_2^{2,0}$, $F_2^{1,1}$, $F_2^{0,2}$ are finite groups of bounded order. But since \mathbb{F}_p^* itself is finite cyclic of bounded order, it suffices to show that the three cohomologies $H^0, H^1, H^2(1 + (p), (\mathbb{Z}/p^n(2)))$ are finite groups of bounded order.

4.5. Let C be a finite cyclic group with generator σ , consider the elements $1 - \sigma$, $N := \sum_{\tau \in C} \tau$ of the group algebra $\mathbb{Z}[C]$, and let A be a $\mathbb{Z}[C]$ -module. Then the sequence

$$0 \rightarrow A \xrightarrow{\sigma-1} A \xrightarrow{N} A \xrightarrow{\sigma-1} A \xrightarrow{N} \dots,$$

in which the first A is in degree zero, forms a complex A^\bullet and

$$H^i(C, A) = H^i A^\bullet.$$

4.6. Returning to the situation and the notation of the proposition, we note that $1 + (p)$ is generated by the element $e^p = 1 + p + \frac{p^2}{2!} + \dots$ and that e^p acts on $\mathbb{Z}/p^n(2)$ by multiplication by e^{2p} . Thus, to complete the proof of the proposition, we need only note that (under our assumption that $p \neq 2$)

$$v_p(e^{2p} - 1) = 1,$$

so that the endomorphism of \mathbb{Z}/p^n given by multiplication by $e^{2p} - 1$ has kernel (p^{n-1}) and cokernel \mathbb{F}_p , both of which have order p , hence in particular bounded, as hoped.

4.7. For the remainder of the section we focus our attention on the terms

$${}^m E_2^{1,1} = H^1((\mathbb{Z}/p^n)^*; H^1(G_{T,n}; \mathbb{Z}/p^m(2))).$$

We again set $n = m$ for simplicity.

4.8. We denote the group $\mu_{p^n}(\mathbb{Q}(\zeta_{p^n}))$ of $(p^n)^{\text{th}}$ roots of 1 in $\mathbb{Q}(\zeta_{p^n})$ by μ_{p^n} for short. μ_{p^n} is isomorphic to $\mathbb{Z}/p^n(2)$ as a \mathbb{Z}/p^n -module. Moreover, if we let an arbitrary element α of $(\mathbb{Z}/p^n)^*$ act on an arbitrary element ζ of μ_{p^n} by ζ^{α^2} , then any such isomorphism becomes equivariant with the action of $(\mathbb{Z}/p^n)^*$. Recalling our formula for the action of a quotient group on the first cohomology of the kernel in terms of cocycles (4.1), and noting that $G_{T,n}$ acts trivially on $\mathbb{Z}/p^n(2)$, we obtain an isomorphism

$$H^1(G_{T,n}, \mathbb{Z}/p^n(2)) \cong \text{Hom}(G_{T,n}, \mu_{p^n})$$

which is not canonical, but is nevertheless equivariant for the action of $(\mathbb{Z}/p^n)^*$ on $\text{Hom}(G_{T,n}, \mu_{p^n})$ given in terms of a continuous map

$$\phi : G_{T,n} \rightarrow \mu_{p^n},$$

an $\eta \in G_{T,n}$, an $\alpha \in (\mathbb{Z}/p^n)^*$, and a lifting γ of α to G_T , by the formula

$$\phi^\alpha(\eta) = (\phi(\gamma\eta\gamma^{-1}))^{\alpha^{-2}}.$$

4.9. Let

$$E = \{a \in \mathbb{Q}(\zeta_{p^n})^* \mid v(a) \equiv 0 \pmod{p^n} \quad \forall v \nmid T\}.$$

Given an element $\alpha \in (\mathbb{Z}/p^n)^*$, and an element $a \in E/\mathbb{Q}(\zeta_{p^n})^{*p^n}$, we let α act on a by

$$\alpha^{-1}(a)^{\alpha^{-1}}.$$

Here, the α^{-1} on the left acts on a through the Galois action of $(\mathbb{Z}/p^n)^*$ on $\mathbb{Q}(\zeta_{p^n})$, while the α^{-1} in the exponent (which may equivalently be put inside the parentheses) denotes multiplication of the base by itself “ α^{-1} many times”, an operation which is only well defined modulo $\mathbb{Q}(\zeta_{p^n})^{*p^n}$.

4.10. An element $a \in E$ gives rise to a Kummer cocycle κ_a , which is unramified outside T . This means that κ_a defines a map $G_{T,n} \rightarrow \mu_{p^n}$ given in terms of an element $\eta \in G_{T,n}$ and a $(p^n)^{\text{th}}$ root a^{1/p^n} of a , by the formula

$$\kappa_a(\eta) = \frac{\eta(a^{1/p^n})}{a^{1/p^n}}.$$

4.11. **Proposition.** In the notation and the situation of paragraphs 4.9 and 4.10, the assignment

$$a \mapsto \kappa_a$$

defines a $(\mathbb{Z}/p^n)^*$ -equivariant isomorphism

$$E/\mathbb{Q}(\zeta_{p^n})^{*p^n} \xrightarrow{\cong} \text{Hom}(G_{T,n}, \mu_{p^n}).$$

The proof is in segments 4.12–4.13.

4.12. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}(\zeta_{p^n})$, let H denote the Galois group of $\overline{\mathbb{Q}}$ over $\mathbb{Q}(\zeta_{p^n})$ and let N denote the Galois group of K_T (3.1) over $\mathbb{Q}(\zeta_{p^n})$:

$$\begin{array}{c} \overline{\mathbb{Q}} \\ \downarrow N \\ K_T \\ \downarrow G_{T,n} \\ \mathbb{Q}(\zeta_{p^n}) \end{array} \quad \left. \vphantom{\begin{array}{c} \overline{\mathbb{Q}} \\ \downarrow N \\ K_T \\ \downarrow G_{T,n} \\ \mathbb{Q}(\zeta_{p^n}) \end{array}} \right\} H$$

Evaluating the Kummer exact sequence

$$1 \rightarrow \mu_{p^n} \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1$$

on $\overline{\mathbb{Q}}$, applying invariants with respect to the action of H , recalling Hilbert’s theorem 90, and noting that H acts trivially on μ_{p^n} , we obtain an isomorphism

$$\kappa : \mathbb{Q}(\zeta_{p^n})^*/\mathbb{Q}(\zeta_{p^n})^{*p^n} \xrightarrow{\cong} \text{Hom}(H, \mu_{p^n}).$$

Then

$$\kappa^{-1}(\text{Hom}(G_{T,n}, \mu_{p^n})) = E/\mathbb{Q}(\zeta_{p^n})^{*p^n}.$$

Indeed, given $a \in E$, κ_a factors through $G_{T,n}$ if and only if

$$\eta(a^{1/p^n}) = a^{1/p^n}$$

for all $\eta \in N$, if and only if

$$\mathbb{Q}(\zeta_{p^n})(a^{1/p^n}) \subset K_T,$$

if and only if $\mathbb{Q}(\zeta_{p^n})(a^{1/p^n})$ is unramified outside T , if and only if

$$v(a) \equiv 0 \pmod{p^n} \quad \forall v \nmid T.$$

4.13. It remains to verify that the map κ is equivariant with respect to the action of $(\mathbb{Z}/p^n)^*$. To this end, fix $\alpha \in (\mathbb{Z}/p^n)^*$, $a \in E$, $\eta \in G_{T,n}$, and a $\gamma \in G_T$ mapping to η . Then we have

$$\begin{aligned}
 \kappa_a^\alpha(\eta) &= (\kappa_a(\gamma\eta\gamma^{-1}))^{\alpha^{-2}} \\
 &= \left(\frac{\gamma\eta\gamma^{-1}(a^{1/p^n})}{a^{1/p^n}} \right)^{\alpha^{-2}} \\
 &= \left(\gamma \frac{\eta(\gamma^{-1}a)^{1/p^n}}{(\gamma^{-1}a)^{1/p^n}} \right)^{\alpha^{-2}} \\
 &= \left(\frac{\eta(\alpha^{-1}a)^{1/p^n}}{(\alpha^{-1}a)^{1/p^n}} \right)^{\alpha^{-1}} \\
 &= \frac{\eta(\alpha^{-1}a^{\alpha^{-1}})^{1/p^n}}{(\alpha^{-1}a^{\alpha^{-1}})^{1/p^n}} \\
 &= \kappa_{\alpha^{-1}a^{\alpha^{-1}}}(\eta) ,
 \end{aligned}$$

indeed.

4.14. **Proposition.** Let \tilde{T} denote the set of primes of $\mathbb{Z}[\zeta_{p^n}]$ above T . We identify \tilde{T} with the set of corresponding valuations of $\mathbb{Q}(\zeta_{p^n})$. Given $\alpha \in (\mathbb{Z}/p^n)^*$ and $b \in (\mathbb{Z}/p^n)^{\tilde{T}}$, we let α act on b by

$$(\alpha \star b)_v = \alpha^{-1} b_{\alpha^{-1}(v)} .$$

Then the formula

$$a \mapsto (v(a))_{v \in \tilde{T}}$$

defines a $(\mathbb{Z}/p^n)^*$ -equivariant isomorphism

$$\frac{E}{\mathbb{Q}(\zeta_{p^n})^{*p^n} \mathbb{Z}[\zeta_{p^n}]^*} \rightarrow (\mathbb{Z}/p^n)^{\tilde{T}} .$$

Proof. By [Was, Corollary 10.5], the p -part of the Picard group of $\mathbb{Z}[\zeta_{p^n}]$ vanishes, so multiplication by p^n on $\text{Pic } \mathbb{Z}[\zeta_{p^n}]$ is an automorphism. Evaluating the short exact sequence of sheaves

$$0 \rightarrow \mathcal{O}^* \rightarrow \mathcal{K}^* \rightarrow \text{Div} \rightarrow 0 ,$$

together with the endomorphism given by multiplication by p^n , on $\mathbb{Z}[\zeta_{p^n}]$ (and recalling that on an integral scheme, \mathcal{K}^* is flasque), we obtain the following diagram

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & & \downarrow \\
 0 & \longrightarrow & \mathbb{Q}(\zeta_{p^n})^*/\mathbb{Z}[\zeta_{p^n}]^* & \longrightarrow & \text{Div } \mathbb{Z}[\zeta_{p^n}] & \longrightarrow & \text{Pic } \mathbb{Z}[\zeta_{p^n}] \longrightarrow 0 \\
 & & \downarrow p^n & & \downarrow p^n & & \downarrow p^n \\
 0 & \longrightarrow & \mathbb{Q}(\zeta_{p^n})^*/\mathbb{Z}[\zeta_{p^n}]^* & \longrightarrow & \text{Div } \mathbb{Z}[\zeta_{p^n}] & \longrightarrow & \text{Pic } \mathbb{Z}[\zeta_{p^n}] \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{Q}(\zeta_{p^n})^*/\mathbb{Z}[\zeta_{p^n}]^* \mathbb{Q}(\zeta_{p^n})^{*p^n} & & (\mathbb{Z}/p^n)^{|\mathbb{Z}[\zeta_{p^n}]|_0} & & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

in which all rows and columns are exact. Here $|\mathbb{Z}[\zeta_{p^n}]|_0$ denotes the set of (nonzero) primes of $\mathbb{Z}[\zeta_{p^n}]$ (in this notation, $\text{Div } \mathbb{Z}[\zeta_{p^n}] = \mathbb{Z}^{|\mathbb{Z}[\zeta_{p^n}]|_0}$). The snake lemma produces an isomorphism

$$\mathbb{Q}(\zeta_{p^n})^* / \mathbb{Z}[\zeta_{p^n}]^* \mathbb{Q}(\zeta_{p^n})^{*p^n} \xrightarrow{\cong} (\mathbb{Z}/p^n)^{|\mathbb{Z}[\zeta_{p^n}]|_0}.$$

It is clear now that the preimage of $(\mathbb{Z}/p^n)^{\tilde{T}}$ is as stated in the theorem.

Regarding equivariance, we note that if K/k is a Galois extension, $a \in K$, α is an automorphism of K/k , and v is a place of K , then α induces an isomorphism

$$K_v \xrightarrow{\cong} K_{\alpha(v)},$$

so $v(\alpha(a)) = \alpha(v)(a)$. This completes the proof of the proposition. \square

4.15. The sequence

$$0 \rightarrow \mu_{p^n} \rightarrow \mathbb{Z}[\zeta_{p^n}]^* \xrightarrow{p^n} \mathbb{Z}[\zeta_{p^n}]^* \rightarrow \frac{E}{\mathbb{Q}(\zeta_{p^n})^{*p^n}} \rightarrow \frac{E}{\mathbb{Z}[\zeta_{p^n}]^* \mathbb{Q}(\zeta_{p^n})^{*p^n}} \rightarrow 0$$

is exact.

This is clear.

4.16. Summarizing, we have the following diagram of $(\mathbb{Z}/p^n)^*$ -modules, in which the vertical sequence is exact.

$$\begin{array}{ccc} 0 & & \\ \downarrow & & \\ \frac{\mathbb{Z}[\zeta_{p^n}]^*}{\mathbb{Z}[\zeta_{p^n}]^{*p^n}} & & \\ \downarrow & & \\ \frac{E}{\mathbb{Q}(\zeta_{p^n})^{*p^n}} & \xrightarrow{\cong} & \text{Hom}(G_{T,n}, \mu_{p^n}) \\ \downarrow & & \\ (\mathbb{Z}/p^n)^{\tilde{T}} & & \\ \downarrow & & \\ 0 & & \end{array}$$

We end our study of the terms ${}_n E_2^{1,1}$ with a discussion of the structure of

$$\frac{\mathbb{Z}[\zeta_{p^n}]^*}{\mathbb{Z}[\zeta_{p^n}]^{*p^n}}$$

as $(\mathbb{Z}/p^n)^*$ -module.

4.17. **Proposition.** Denote ζ_{p^n} by ζ for short. For $a \in (\mathbb{Z}/p^n)^*$, let

$$\xi_a = \zeta^{\frac{1-a}{2}} \frac{1 - \zeta^a}{1 - \zeta}.$$

Then we have

$$(1) \quad \xi_1 = 1,$$

and for each $a \in (\mathbb{Z}/p^n)^*$,

$$(2) \quad \xi_a \equiv \xi_{-a} \pmod{\mathbb{Z}[\zeta_{p^n}]^{*p^n}}.$$

The elements ξ_a of $\mathbb{Z}[\zeta_{p^n}]^*/\mathbb{Z}[\zeta_{p^n}]^{*p^n}$ parametrized by

$$a \in (\mathbb{Z}/p^n)^*/\langle -1 \rangle$$

are free except for the single relation (1). If B denotes the \mathbb{Z}/p^n -submodule generated by these elements, then

$$\mathbb{Z}[\zeta_{p^n}]^*/\mathbb{Z}[\zeta_{p^n}]^{*p^n} = \mu_{p^n} \oplus B.$$

The proof is in paragraphs 4.18–4.19.

4.18. Equation (1) is clear. To verify (2), we carry out the following computation inside $\mathbb{Z}[\zeta_{p^n}]^*$:

$$\begin{aligned} \xi_{-a} &= \zeta^{\frac{1+a}{2}} \cdot \frac{\zeta^a}{\zeta^a} \cdot \frac{1 - \zeta^{-a}}{1 - \zeta^a} \cdot \frac{1 - \zeta^a}{1 - \zeta} \\ &= \zeta^{\frac{1+a}{2}} \cdot \frac{-1}{\zeta^a} \cdot \frac{1 - \zeta^a}{1 - \zeta} \\ &= -\xi_a, \end{aligned}$$

and note that

$$-1 = (-1)^{p^n} \equiv 1 \pmod{\mathbb{Z}[\zeta_{p^n}]^{*p^n}}.$$

4.19. Let $U := \mathbb{Z}[\zeta_{p^n}]^*$, let C^+ denote the subgroup generated by the elements ξ_a , $a \in (\mathbb{Z}/p^n)^*$, and let U^+ denote the subgroup of U of totally real units. Then by [Was, Theorem 8.2], C^+ is a subgroup of U^+ of index h^+ , the class number of the maximal totally real subfield. By [Was, Theorem 4.12], $\mu_{p^n} \oplus U^+$ has index 1 or 2 in U . Since $h^+|h$ and h is coprime to p , it follows that $\mu_{p^n} \oplus C^+ \leq U$ is a subgroup of finite index coprime to p . According to the Dirichlet unit theorem,

$$U \cong \mu_{p^n} \oplus \mathbb{Z}^{r+s-1}$$

where r is the number of real places, and s is the number of complex conjugate pairs of complex places. Thus, in our case, $r = 0$ and

$$s = |(\mathbb{Z}/p^n)^*/\langle -1 \rangle|.$$

It follows that the ξ_a generate a free abelian group of rank $s - 1$, and that their image modulo U^{p^n} , together with μ_{p^n} , generates all of U/U^{p^n} . This completes the proof of the proposition.

4.20. Let $(\mathbb{Z}/p^n)^*$ act on $\mathbb{Z}[\zeta_{p^n}]^*/\mathbb{Z}[\zeta_{p^n}]^{*p^n}$ by $\beta \circ a = \beta(a)^\beta$. This is the action induced by the action defined in paragraph 4.9, except for having taken the liberty to precompose with the automorphism of $(\mathbb{Z}/p^n)^*$ given by $\alpha \mapsto \alpha^{-1}$. We recall that here multiplication by β on the left refers to the Galois action, while the exponent refers to multiplication inside $\mathbb{Z}[\zeta_{p^n}]^*$.

4.21. **Proposition.** We have

$$(3) \quad \beta \circ \zeta = \zeta^{\beta^2}$$

for any $\zeta \in \mu_{p^n}$, and

$$(4) \quad \beta \circ \xi_a = \xi_{\beta a}^\beta \xi_\beta^{-\beta}.$$

Proof. Equation (3) is clear. To verify (4), we compute, focusing on the Galois action:

$$\begin{aligned} \beta(\xi_a) &= \zeta^{\beta \frac{1+a}{2}} \frac{1 - \zeta^{\beta a}}{1 - \zeta^\beta} \\ &= \zeta^{\frac{1-\beta a}{2} - \frac{1-\beta}{2}} \cdot \frac{1 - \zeta^{\beta a}}{1 - \zeta} \cdot \left(\frac{1 - \zeta^\beta}{1 - \zeta} \right)^{-1} \\ &= \xi_{\beta a} \xi_\beta^{-1}. \end{aligned}$$

□

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